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Three-Dimensional Vertex Model in Statistical Mechanics from Baxter–Bazhanov Model

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We find that the Boltzmann weight of the three-dimensional Baxter–Bazhanov model is dependent on four spin variables which are the linear combinations of the spins on the corner sites of the cube, and the Wu–Kadanoff–Wegner duality between the cube- and vertex-type tetrahedron equations is obtained explicitly for the Baxter–Bazhanov model. Then a three-dimensional vertex model is obtained by considering the symmetry property of the weight function, which corresponds to the three-dimensional Baxter–Bazhanov model. The vertex-type weight function is parametrized as the dihedral angles between the rapidity planes connected with the cube. We write down the symmetry relations of the weight functions under the actions of the symmetry group G of the cube. The six angles with a constraint condition appearing in the tetrahedron equation can be regarded as the six spectra connected with the six spaces in which the vertex-type tetrahedron equation is defined.

KEY WORDS: Three-dimensional Baxter–Bazhanov model; interaction-round-a-cube (IRC) model; Wu–Kadanoff–Wegner duality; cube-type tetrahedron equation; vertex-type tetrahedron equation; vertex-type tetrahedron equation; vertex-type tetrahedron equation; vertex-type model; three-dimensional lattice model; symmetry property; three-dimensional vertex model; spherical trigonometry; vertex-type weight function; three-dimensional star–star relation; additional constraints; dihedral angles; symmetry group.

1. INTRODUCTION

Three-dimensional integrable models in statistical mechanics have attracted much attention recently. As a factorized scattering theory in $2+1$ dimensions, Zamolodchikov's model was formulated in 1980[1] with $N=2$, where

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he conjectured that it satisfied the tetrahedron equation, which is also the condition that the transfer matrices of the three-dimensional lattice models commute in statistical mechanics.[2] This equation was verified by Baxter in 1983.[3] The Wu–Kadanoff–Wegner duality[4,5] found in the early 1970s was applied first to this lattice model by Baxter for $N=2$, who commented on some subtleties with the above duality. In 1992, Bazhanov and Baxter[6] generalized the two-state Zamolodchikov model to arbitrary states. We call this the Baxter–Bazhanov model. It is an interaction-round-a-cube model with $N \geqslant 2$. Kashaev *et al.*[7] showed that the Boltzmann weights of the Baxter–Bazhanov model satisfy the cube-type tetrahedron equation, by introducing the star–square relation for which a connection was found[8] with the chiral Potts model. The restricted star–triangle relation and the star–star relation of this model have been discussed in detail in refs. 9–12. They connected with the quantum dilogarithm[13] and the shift operator in the discrete space-time picture.[14,15]

A new series of three-dimensional integrable lattice models was presented by Mangazeev *et al.*,[16] the weight functions of which satisfy the modified tetrahedron equation.[17] Recently, Cerchiai *et al.*[50] studied the Baxter–Bazhanov model from the point of view of link theory and gave the representations of the braid group if some suitable spectral limits are taken.

Korepanov[18] obtained the solution of the vertex tetrahedron equation with the spin variables taking $N=2$, which leads to a commuting family of transfermatrices. With respect to the scattering process, Hietarinta discussed the three corresponding tetrahedron equations in which the Frenkel–Moore equation was fitted[19,20] and proposed another vertex solution with 16 nonzero weights.[21] The discrete symmetry groups of vertex models were studied by Boukraa *et al.*[51] As a generalization of Hietarinta's solution of the tetrahedron equation Mangazeev *et al.*[22] proposed another N-state spin integrable model on a three-dimensional lattice and this model can be reformulated as a vertex model. The weight function of this model can be obtained from the Baxter–Bazhanov model by taking appropriate limits.[23] In principle, many of the solved problems can be formulated as vertex models and the interaction-round-a-cube (IRC) model is always equivalent to an N^4-state vertex model by using a straightforward generalization of the arguments of Perk and Wu.[24] Au-Yang and Perk[25] claim this equivalence to be one of the basic ingredients in the original construction of the integrable chiral Potts model. This mapping is a one-to-one mapping and preserves all integrability conditions trivially at the expense of numerous zero vertex weights. In this paper, the more special Wu–Kadanoff–Wegner equivalence explicitly uses four Z_N symmetries of the model, but is not one-to-one. However, the resulting vertex weights are simpler than the IRC weights. Even though there are subleties due to the N^4-to-1 mapping, all integrability conditions are kept. It appears that the Protvino group had some oversights in their earlier work, but their new work[26] and the current work independently come to similar conclusions.

This paper is organized as follows. In Section 2 we give a brief description of the Baxter–Bazhanov model and the duality between the cube- and the vertex-type tetrahadron equations. The weight functions of the Baxter–Bazhanov model are written as vertex forms in Section 3 and some symmetry properties are given for this three-dimensional model. Then the duality is obtained explicitly for the Baxter–Bazhanov model. By using the symmetry properties of the weight functions we get the vertex-type weight functions for the three-dimensional vertex model. It should be noted that the weight functions of the model proposed by Mangazeev *et al.* can be obtained from these vertex-type weight functions when we take the limit of the spectrum and use the star–triangle relation of the Baxter–Bazhanov model. In Section 4 the vertex-type weight functions are parametrized as the angles of spherical triangles by using the methods of spherical trigonometry parametrization. These angles are the dihedral angles between the "rapidity planes" passing the cubes, as in the Zamolochikov model. In this way, the spectra appearing in the vertex-type tetrahedron equation can be denoted by these angles and they connect with the spaces in which the vertex-type tetrahedron equation is defined. In Section 5 we discuss the constraint conditions imposed on the tetrahedron equations from the point of view of angle variables. Then the symmetry properties of the vertex-type weight functions are discussed. They are symmetrical about the transformations of the group G consisting of various rotations, reflections, and their combinations of the cube. Finally, some conclusions and remarks are given.

2. BAXTER–BAZHANOV MODEL AND DUALITY BETWEEN CUBE- AND VERTEX-TYPE TETRAHEDRON EQUATIONS

2.1. Three-Dimensional Baxter–Bazhanov Model

As is well known, the Baxter–Bazhanov model is an interaction-round-a-cube (IRC) model. Its partition function reads

$$Z = \sum_{\text{spins}} \prod_{\text{cubes}} W(a \mid efg \mid bcd \mid h) \tag{1}$$

where $W(a \mid efg \mid bcd \mid h)$ is the Boltzmann weight of the spin configuration $a, \ldots, h$ (see Fig. 1) and these spin variables take their values in Z_N with

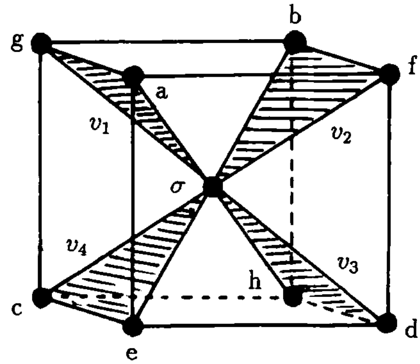

Fig. 1. Arrangements of the spins $a, \ldots, h$ on the corner sites and the spin σ in the center of an elementary cube of the simple cubic lattice $\mathscr{L}$.

$N \geqslant 2$. The product is over all elementary cubes in the simple cubic lattice $\mathscr{L}$. The Boltzmann weight $W(a \mid efg \mid bcd \mid h)$ can be written as

$$W(a \mid efg \mid bcd \mid h)$$
$$= \frac{w(v_4/v_3, e-c-d+h)\, s(g, a-g-f+b)}{w(v_4/v_3, a-g-f+b)\, s(c-h, h-d)}$$
$$\times \left\{ \sum_{\sigma=0}^{N-1} \frac{w(v_2, b-f+\sigma)\, w(v_3, d-h-\sigma)\, s(\sigma, a)\, s(\sigma, h)}{w(v_1, g-a+\sigma)\, w(v_4, e-c-\sigma)\, s(\sigma, c)\, s(\sigma, f)} \right\}_0 \tag{2}$$

with the relation $\omega v_1 v_4 = v_2 v_3$; the subscript 0 on the curly brackets indicates that the expression inside is divided by itself with the zero exterior spins, and we have used the notations

$$\frac{w(v, a)}{w(v, 0)} = [\Delta(v)]^a \prod_{j=1}^{a} (1 - \omega^j v)^{-1}, \qquad \Delta(v) = (1 - v^N)^{1/N} \tag{3}$$

$$\omega = \exp(2\pi i/N), \qquad \omega^{1/2} = \exp(\pi i/N), \qquad s(a, b) = \omega^{ab} \tag{4}$$

Note that the Boltzmann weight function (2) describes a very special type of interaction of eight spins around the cube as in Fig. 1. There are three-spin interactions on the triangles (a, g, σ), (b, f, σ), (d, h, σ), and (c, e, σ), described by $w(v, a)$ or by $1/w(v, a)$, and two-spin interactions $s(\sigma, a)$, $s(\sigma, c)$, $s(\sigma, h)$, and $s(\sigma, f)$ associated with the edges linking σ to a, c, f, and h in the curly brackets. The factors before the curly brackets denote the spin interactions in the planes (a, f, b, g) and (c, e, d, h). After introducing an overall normalization and some additional multipliers,[10] we express the weight function of the Baxter–Bazhanov model as

$$
\begin{aligned}
&W_P(a \mid efg \mid bcd \mid h)\\
&\quad=\frac{\omega^{fb}}{\omega^{ag}}\left[\frac{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid a+d, e+f)}{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid g+h, c+b)}\right]^{1/2}\\
&\qquad\times\left[\frac{w(x_4, x_{34}, x_3 \mid e+h, d+c)}{w(x_4, x_{34}, x_3 \mid a+b, f+g)}\right]^{1/2}\\
&\qquad\times\left[\frac{w(x_2, x_{12}, x_1 \mid e+g, a+c)}{w(x_2, x_{12}, x_1 \mid d+b, f+h)}\right]^{1/2}\frac{\omega^{(ag+gb+bh)/2}}{\omega^{(hd+de+ea)/2}}\\
&\qquad\times\left\{\sum_{\sigma\in Z_N}\frac{w(x_3, x_{13}, x_1 \mid d, h+\sigma)\, w(x_4, x_{24}, x_2 \mid a, g+\sigma)}{w(x_4, x_{14}, x_1 \mid e, c+\sigma)\, w(x_3/\omega, x_{23}, x_2 \mid f, b+\sigma)}\right\}_0
\end{aligned}
\tag{5}
$$

It satisfies the tetrahedron equation, which ensures the commutativity of the layer-to-layer transfer matrices. Here we have used

$$
\begin{aligned}
w(x, y, z \mid k, l) &= w(x, y, z \mid k-l)\, \Phi(l)\\
w(x, y, z \mid l) &= \prod_{j=1}^{l} \frac{y}{z - x\omega^j}, \qquad k, l \in Z_N
\end{aligned}
\tag{6}
$$

with the notation

$$
x^N + y^N = z^N, \qquad \Phi(l) = \omega^{l(l+N)/2}, \qquad x_i^N - x_j^N = x_{ij}^N \tag{7}
$$

for $i<j$ and $i, j = 1, 2, 3, 4$.

2.2. Duality Between Cube- and Vertex-Type Tetrahedron Equations

The Boltzmann weight function W in Eq. (5) satisfies the following tetrahedron equation[7]:

$$
\begin{aligned}
&\sum_d W(a_4 \mid c_2c_1c_3 \mid b_1b_3b_2 \mid d)\, W'(c_1 \mid b_2a_3b_1 \mid c_4dc_6 \mid b_4)\\
&\qquad\times W''(b_1 \mid dc_4c_3 \mid a_2b_3b_4 \mid c_5)\, W'''(d \mid b_2b_4b_3 \mid c_5c_2c_6 \mid a_1)\\
&\quad=\sum_d W'''(b_1 \mid c_1c_4c_3 \mid a_2a_4a_3 \mid d)\, W''(c_1 \mid b_2a_3a_4 \mid dc_2c_6 \mid a_1)\\
&\qquad\times W'(a_4 \mid c_2dc_3 \mid a_2b_3a_1 \mid c_5)\, W(d \mid a_1a_3a_2 \mid c_4c_5c_6 \mid b_4)
\end{aligned}
\tag{8}
$$

where W, W', W'', and W''' are four sets of Boltzmann weights. With respect to the scattering process and using particle labeling schemes, Hietarinta[21] wrote down the vertex-type tetrahedron equation[18,22]

$$\sum_{\substack{k_1, k_2, k_3,\\ k_4, k_5, k_6}} R^{k_1, k_2, k_3}_{i_1, i_2, i_3} R'^{j_1 k_4 k_5}_{k_1 i_4 i_5} R''^{j_2 j_4 k_6}_{k_2 k_4 i_6} R'''^{j_3 j_5 j_6}_{k_3 k_5 k_6}$$

$$= \sum_{\substack{k_1, k_2, k_3,\\ k_4, k_5, k_6}} R'''^{k_3, k_5, k_6}_{i_3, i_5, i_6} R''^{k_2 k_4 j_6}_{i_2 i_4 k_6} R'^{k_1 j_4 j_5}_{i_1 k_4 k_5} R^{j_1 j_2 j_3}_{k_1 k_2 k_3} \tag{9}$$

We call relation (8) the cube-type tetrahedron equation. Just as with the Wu–Kadanoff–Wegner duality in the Yang–Baxter equation, the tetrahedron analog of the Wu–Kadanoff–Wegner duality between the above two types of tetrahedron equations can be constructed by

$$W(a \mid efg \mid bcd \mid h) = R^{\alpha d + \beta h + \gamma b + \delta f,\, \alpha h + \beta c + \gamma g + \delta b,\, \alpha d + \beta e + \gamma c + \delta h}_{\alpha e + \beta c + \gamma g + \delta a,\, \alpha d + \beta e + \gamma a + \delta f,\, \alpha f + \beta a + \gamma g + \delta b} \tag{10}$$

where the constants $\alpha, \beta, \gamma, \delta$ are the parameters of the map F_{cv}: $W = R \circ F_{cv}$.[21] There are two nontrivial results about the map F_{cv}:

$$R^{lmn}_{ijk} = 0 \qquad \text{unless} \quad \alpha l + \beta i = \beta m + \alpha j \quad \text{and} \quad \gamma m + \beta j = \beta n + \gamma k \tag{11}$$

for the case of $\alpha\gamma = \beta\delta$, and

$$R^{lmn}_{ijk} = 0 \qquad \text{unless} \quad m = i + k \quad \text{and} \quad j = l + n \tag{12}$$

for the case of $\alpha = \gamma = 0$ and $\beta = -\delta = 1$. The solution presented in ref. 22 corresponds to the latter, which can be obtained from the Boltzmann weight of the Baxter–Bazhanov model by taking appropriate limits.[23] In the following section the map F_{cv} will be obtained for the three-dimensional Baxter–Bazhanov model. Then we get a three-dimensional vertex model which corresponds to the IRC model.

3. THREE-DIMENSIONAL VERTEX MODEL

In this section, some symmetry properties are found for the weight functions of the Baxter–Bazhanov model. We can interpret these properties as generalizations of the symmetry properties of the weight functions of the $N = 2$ Zamolodchikov model proposed by Baxter.[3] Then we get that the weight functions of Baxter–Bazhanov model are vertex-type weight functions and a three-dimensional vertex model is constructed. The map F_{cv} is given explicitly for this three-dimensional model. Six spectra exist in the vertex-type tetrahedron equation. They are the spectra connected with the six spaces in which the vertex-type tetrahedron equation is defined.

3.1. The Vertex-Type Boltzmann Weight

Baxter[3] discussed in detail the duality between the Zamolodchikov plaquettes for the two-color model of the straight-string scattering theory[1] and the elementary cube with spins $a, b,..., h$ taking their values ± 1 on the corner sites for the IRC model in statistical mechanics. He wrote down the weight function $W(a \mid efg \mid bcd \mid h)$ of the $N=2$ Zamolodchikov model as

$$W(a \mid efg \mid bcd \mid h) = S \begin{matrix} cg, & ae, & df, & bh \\ de, & af, & bg, & ch \\ bf, & ag, & ce, & dh \end{matrix} \tag{13}$$

where S is the three-string scattering amplitude. Baxter proved that it satisfies the tetrahedron equation, which is the factorizable condition of the $(2+1)$-dimensional scattering theory in field theory and the integrable condition of the three-dimensional lattice model in statistical mechanics. The weight function $W(a \mid efg \mid bcd \mid h)$ has the property

$$W(a \mid efg \mid bcd \mid h) = W(-a \mid -e, -f, -g \mid -b, -c, -d \mid -h) \tag{14}$$

Notice that the spins $a,..., h$ take values ± 1 in the above two relations. We know that the Baxter–Bazhanov model is the generalization of the $N=2$ Zamolodchikov model. Then a natural problem occurs. What is the generalization of the property (14) in the Baxter–Bazhanov model? By considering the relations (6) and (7), we find that the weight function $W(a \mid efg \mid bcd \mid h)$ of the Baxter–Bazhanov model, given in expression (5), has the symmetry properties

$$W(a \mid efg \mid bcd \mid h) = W(a \pm 1 \mid e \pm 1, f \pm 1, g \mid b, c, d \pm 1 \mid h) \tag{15}$$

$$W(a \mid efg \mid bcd \mid h) = W(a \mid e, f, g \pm 1 \mid b \pm 1, c \pm 1, d \mid h \pm 1) \tag{16}$$

They correspond to the property (14) of the Zamolodchikov model when $N=2$. Owing to these symmetry properties, the N^8 weight functions $W(a \mid efg \mid bcd \mid h)$ of the Baxter–Bazhanov model reduce to N^6 ones with the degeneracy being N^2. This provides us with the possibility to construct the three-dimensional vertex model from the Baxter–Bazhanov model due to the vertex-type Boltzmann weights mapping as R: $I_v^{\otimes 6} \to \mathscr{C}$ in three dimensions.

From expressions (2) and (5), we know that the Boltzmann weights of the Baxter–Bazhanov model map as $W: I_c^{\otimes 8} \to \mathscr{C}$, and the cube-type tetrahedron equation is defined on $I_c^{\otimes 15}$. The relations (11) and (12) mean that two labels of the vertex-type weight function R are determined from the other ones. What happens in the three-dimensional Baxter–Bazhanov model? We deal with this now. Set

$$r_1 = d-h, \quad r_2 = a-g, \quad r_3 = e-c, \quad r_4 = f-b, \quad r_5 = g+h-b-c \tag{17}$$

Using relation (6), we can change expression (5) into the form

$$\begin{aligned}
&W_P(a \mid efg \mid bcd \mid h) \\
&\quad = I_\omega \left[\frac{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid r_1+r_2-r_3-r_4+r_5)}{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid r_5)} \right. \\
&\qquad \times \left. \frac{w(x_4, x_{34}, x_3 \mid r_3-r_1)\, w(x_2, x_{12}, x_1 \mid x_3-r_2)}{w(x_4, x_{34}, x_3 \mid r_2-r_4)\, w(x_2, x_{12}, x_1 \mid r_1-r_4)} \right]^{1/2} \\
&\qquad \times \left\{ \sum_{\sigma \in Z_N} \frac{w(x_3, x_{13}, x_1 \mid r_1+\sigma)\, w(x_4, x_{24}, x_2 \mid r_2+\sigma)}{\omega^{\sigma r_5} w(x_4, x_{14}, x_1 \mid r_3+\sigma)\, w(x_3/\omega, x_{23}, x_2 \mid r_4+\sigma)} \right\}_0
\end{aligned} \tag{18}$$

where

$$I_\omega = (-)^{r_5} (\omega^{1/2})^{r_3^2 + r_3 r_4 - r_1 r_3 - r_2 r_3 - r_3 r_5 - r_4 r_5} \left[\frac{\Phi(r_1)\, \Phi(r_2)}{\Phi(r_3)\, \Phi(r_4)} \right]^{1/2} \tag{19}$$

$\Phi(r_i)$, $i = 1, 2, 3, 4$, are given in relation (7). By taking account of the property

$$w(x, y, z \mid l)\, w(z, \omega^{1/2} y, \omega x \mid -l)\, \Phi(l) = 1, \qquad l \in Z_N \tag{20}$$

we find for the weight function

$$\begin{aligned}
&W_P(a \mid efg \mid bcd \mid h) \\
&\quad = (-)^{k_2} (\omega^{1/2})^{k_1 k_2 + k_2 k_3 + k_1 k_3} \left[\frac{w(x_1, \omega^{1/2} x_{12}, \omega x_2 \mid k_1)}{w(x_1, \omega^{1/2} x_{12}, \omega x_2 \mid k_4 - k_3)} \right. \\
&\qquad \times \left. \frac{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid k_4 - k_1 - k_2 - k_3)\, w(x_4, x_{34}, x_3 \mid k_3)}{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid -k_2)\, w(x_4, x_{34}, x_3 \mid k_4 - k_1)} \right]^{1/2} \\
&\qquad \times \left\{ \sum_{\sigma \in Z_N} \frac{\omega^{\sigma k_2} w(x_3, x_{13}, x_1 \mid \sigma)\, w(x_4, x_{24}, x_2 \mid k_4+\sigma)}{w(x_4, x_{14}, x_1 \mid k_3+\sigma)\, w(x_3/\omega, x_{23}, x_2 \mid k_1+\sigma)} \right\}_0
\end{aligned} \tag{21}$$

where

$$k_1 = r_4 - r_1, \quad k_2 = -r_5, \quad k_3 = r_3 - r_1, \quad k_4 = r_2 - r_1 \tag{22}$$

and we used the relation

$$
\begin{aligned}
2bf &- ag + gb + bh - dh - de - ea + h^2 - b^2 - c^2 + 2(h-d)(b+c-g-h) \\
&\quad + d^2 + (a+c)(e+g) - (b+d)(f+h) + bd + ef + cd - eg - bc - fg \\
&= -b^2 - bc - bd + bf + bg + 2bh - c^2 - cd + ce + cg + 2ch \\
&\quad + d^2 - de - df + 2dg + ef - eg - fg - 2gh - h^2 \\
&= k_1k_2 + k_1k_3 + k_2k_3
\end{aligned}
\tag{23}
$$

The factor $(\omega^{1/2})^{k_1k_2+k_2k_3+k_1k_3}$ on the RHS terms of relation (21) is dependent on the spin variables k_1, k_2, k_3, which can be called the final states;[1] it is formed of three parts: one is the original factor $(\omega^{1/2})^{gb+bh+2fb-hd-de-ea-ag}$ and the other two come from the Φ factors in the prefactors and the summation. The weight function $W_P(a \mid efg \mid bcd \mid h)$ is dependent on the four spin variables k_1, k_2, k_3, k_4. The related spectral variables x_i, x_{ij} $(1 \leqslant i < j \leqslant 4)$ will be discussed in the next section. Furthermore, the Boltzmann weight of the Baxter–Bazhanov model can be reformulated as

$$
\begin{aligned}
&R^{j_1j_2j_3}_{i_1i_2i_3} \\
&\quad = (-)^{j_2}(\omega^{1/2})^{j_1j_2+j_2j_3+j_1j_3} \\
&\qquad \times \left[\frac{w(x_1, \omega^{1/2}x_{12}, \omega x_2 \mid j_1)\, w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid -i_2)\, w(x_4, x_{34}, x_3 \mid j_3)}{w(x_1, \omega^{1/2}x_{12}, \omega x_2 \mid i_1)\, w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24} \mid -j_2)\, w(x_4, x_{34}, x_3 \mid i_3)}\right]^{1/2} \\
&\qquad \times \left\{\sum_{\sigma \in Z_N} \frac{\omega^{\sigma j_2} w(x_3, x_{13}, x_1 \mid \sigma)\, w(x_4, x_{24}, x_2 \mid i_1 + j_3 + \sigma)}{w(x_4, x_{14}, x_1 \mid j_3 + \sigma)\, w(x_3/\omega, x_{23}, x_2 \mid j_1 + \sigma)}\right\}_0
\end{aligned}
\tag{24}
$$

where the spin variables $i_1, i_2, i_3, j_1, j_2, j_3$ satisfy the conditions $i_1 + i_2 = j_1 + j_2$, $i_2 + i_3 = j_2 + j_3$ and

$$
\begin{aligned}
i_1 &= a + c - e - g, \qquad & i_2 &= e + f - a - d, \qquad & i_3 &= a + b - f - g \\
j_1 &= f + h - b - d, \qquad & j_2 &= b + c - g - h, \qquad & j_3 &= e + h - c - d
\end{aligned}
\tag{25}
$$

In this way, we denote the weight function of the Baxter–Bazhanov model as a vertx weight function and the vertex-type spin variables $i_1, i_2, i_3, j_1, j_2, j_3$ are chosen by (25), which is not arbitrary. Here we regard i_1, i_2, i_3 as the initial states and j_1, j_2, j_3 as the final states in space 1, 2, 3, respectively.[1] Then, in the terms on the LHS of the tetrahedron equation (8), the final states k_1, k_2, k_3 defined by the weight function W are the same as the initial states in the first spaces defined by the weight functions W', W'', W''' with the use of the rules (25), respectively. The final states k_3, k_5, k_6 defined by the weight function W''' are the same as the initial states in the third spaces defined by W, W', W'', respectively, on the RHS terms of the tetrahedron equation (8) by using the rules (25). .... They are just the conditions of the duality between the cube-type tetrahedron equation and the vertex-type tetrahedron equation. Comparing with the relation (10), we known that the parameters of the map F_{cv} are $\alpha=-\beta=\gamma=-\delta=-1$. Of course, we can make the transformations $a\to -a$, $b\to -b,...,$ $h\to -h$ in (25). In this case, we have that $\alpha=-\beta=\gamma=-\delta=1$. The expression (24) can be interpreted as the Boltzmann weight of a three-dimensional vertex model. The Boltzmann weight of the vertex model proposed in ref. 22 can be obtained when we set $i_3=j_3=0$ and use the star–triangle relation of the Baxter–Bazhanov model.[23]

The key point is that we can get the vertex-type weight function (24) due to the symmetry properties (15) and (16). It is surprising that the $\omega^{1/2}$-factors in the weight function of the Baxter–Bazhanov model have the form $\omega^{j_1j_2+j_2j_3+j_1j_3}$. In this way, the spin variables in the weight function of the Baxter–Bazhanov model, including the prefactors, can be changed into the vertex-type spin variables $i_1, i_2, i_3, j_1, j_2, j_3$. And these prefactors are very important for ensuring the weight function symmetry under the action of the group G of all symmetry transformations of a three-dimensional cube[10,9] and were useful when Kashaev *et al.* proved the tetrahedron equation of the Baxter–Bazhanov model. They are also necessary for the symmetrical vertex-type weight function $R^{j_1j_2j_3}_{i_1i_2i_3}$.

3.2. The Spectral Parameters in the Weight Function

From the symmetry properties of the Boltzmann weights of the Baxter–Bazhanov model, we have the relation

$$\left\{\sum_{\sigma\in Z_N}\frac{w(x_3,x_{13},x_1\mid\sigma+a)\,w(x_4,\omega_{24},x_2\mid\sigma+c)\,s(\sigma,n)}{w(x_4,x_{14},x_1\mid\sigma+b)\,w(x_3/\omega,x_{23},x_2\mid\sigma+d)}\right\}_0$$

$$=\frac{w(x_4,x_{34},x_3\mid c-d)}{w(x_4,x_{34},x_3\mid b-a)\,s(a,n)}$$

$$\times\left\{\sum_{\sigma\in Z_N}\frac{w(x_4x_{13},x_1x_{34},x_3x_{14}\mid\sigma-a+b+n)\,w(x_{23},x_{34},x_{24}\mid\sigma)\,s(\sigma,d)}{w(x_{13},\omega x_{34},\omega x_{14}\mid\sigma+n)\,w(x_4x_{23},x_2x_{34},x_3x_{24}\mid\sigma+c-d)\,s(\sigma,a)}\right\}_0 \tag{26}$$

where $a, b, c, d, \sigma, n\in Z_N$.[7,11] By using the above relation we can write the vertex-type weight function (24) as

$$
\begin{aligned}
&R^{j_1 j_2 j_3}_{i_1 i_2 i_3} \\
&= (-)^{j_2} (\omega^{1/2})^{j_1 j_2 + j_2 j_3 + j_1 j_3} \\
&\quad \times \left[\frac{w(x_1, \omega^{1/2} x_{12}, \omega x_2 \mid j_1)\, w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24} \mid -i_2)\, w(x_4, x_{34}, x_3 \mid i_3)}{w(x_1, \omega^{1/2} x_{12}, \omega x_2 \mid i_1)\, w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24} \mid -j_2)\, w(x_4, x_{34}, x_3 \mid j_3)} \right]^{1/2} \\
&\quad \times \left\{ \sum_{\sigma \in Z_N} \frac{w(x_4 x_{13}, x_1 x_{34}, x_3 x_{14} \mid \sigma + j_2 + j_3)\, w(x_{23}, x_{34}, x_{34} \mid \sigma)\, s(\sigma, j_1)}{w(x_{13}, \omega x_{34}, \omega x_{14} \mid \sigma + j_2)\, w(x_4 x_{23}, x_2 x_{34}, x_3 x_{24} \mid \sigma + i_3)} \right\}_0
\end{aligned} \tag{27}
$$

Set

$$
u = \frac{x_1}{\omega x_2}, \qquad v = \frac{x_4}{x_3}, \qquad z = \frac{z_1}{z_2}, \qquad z_1 = \frac{x_{13}}{\omega x_{14}}, \qquad z_2 = \frac{x_{23}}{x_{24}} \tag{28}
$$

The Boltzmann weight of the three-dimensional vertex model shown in Fig. 2 has the form

$$
\begin{aligned}
&R(u, z, v)^{j_1 j_2 j_3}_{i_1 i_2 i_3} \\
&\quad = (-)^{j_2} (\omega^{1/2})^{j_1 j_2 + j_2 j_3 + j_1 j_3} \left[\frac{w(u, j_1)\, w(z_2/(\omega z_1), -i_2)\, w(v, i_3)}{w(u, i_1)\, w(z_2/(\omega z_1), -j_2)\, w(v, j_3)} \right]^{1/2} \\
&\quad\quad \times \left\{ \sum_{\sigma \in Z_N} \frac{w(\omega v z_1, \sigma + j_2 + j_3)\, w(z_2, \sigma)\, s(\sigma, j_1)}{w(z_1, \sigma + j_2)\, w(v z_2, \sigma + i_3)} \right\}_0
\end{aligned} \tag{29}
$$

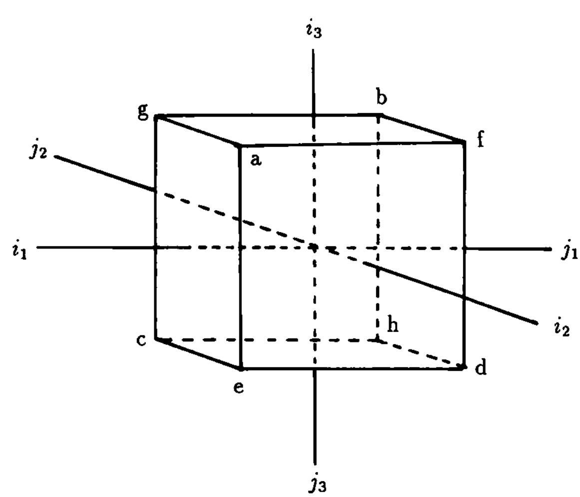

Fig. 2. The Boltzmann weight of the three-dimensional vertex model corresponding to the IRC model.

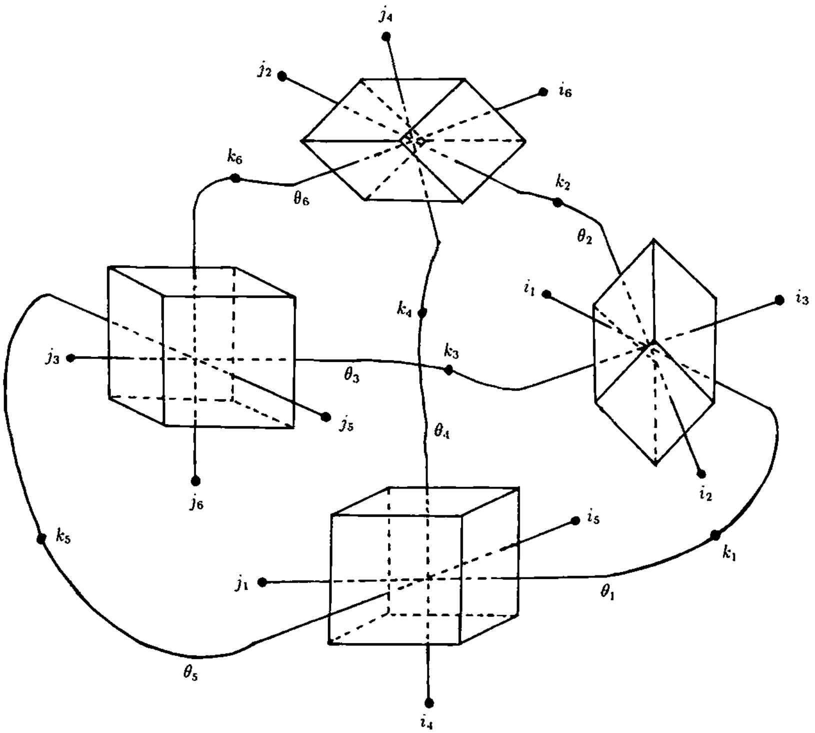

Fig. 3. The left-hand sides of the tetrahedron equations.

where we have used the notation (3). It satisfies the vertex-type tetrahedron equation

$$\begin{aligned}
&\sum_{\substack{\{k_i\},\\ i=1,\dots,6}} R(u_1, u_2, u_3)_{i_1,\, i_2,\, i_3}^{k_1,\, k_2,\, k_3} R(u_1, u_4, u_5)_{k_1 i_4 i_5}^{j_1 k_4 k_5} \\
&\qquad \times R(u_2, u_4, u_6)_{k_2 k_4 i_6}^{j_2 j_4 k_6} R(u_3, u_5, u_6)_{k_3 k_5 k_6}^{j_3 j_5 j_6} \\
&= \sum_{\substack{\{k_i\},\\ i=1,\dots,6}} R(u_3, u_5, u_6)_{i_3,\, i_5,\, i_6}^{k_3,\, k_5,\, k_6} R(u_2, u_4, u_6)_{i_2 i_4 k_6}^{k_2 k_4 j_6} \\
&\qquad \times R(u_1, u_4, u_5)_{i_1 k_4 k_5}^{k_1 j_4 j_5} R(u_1, u_2, u_3)_{k_1 k_2 k_3}^{j_1 j_2 j_3}
\end{aligned} \tag{30}$$

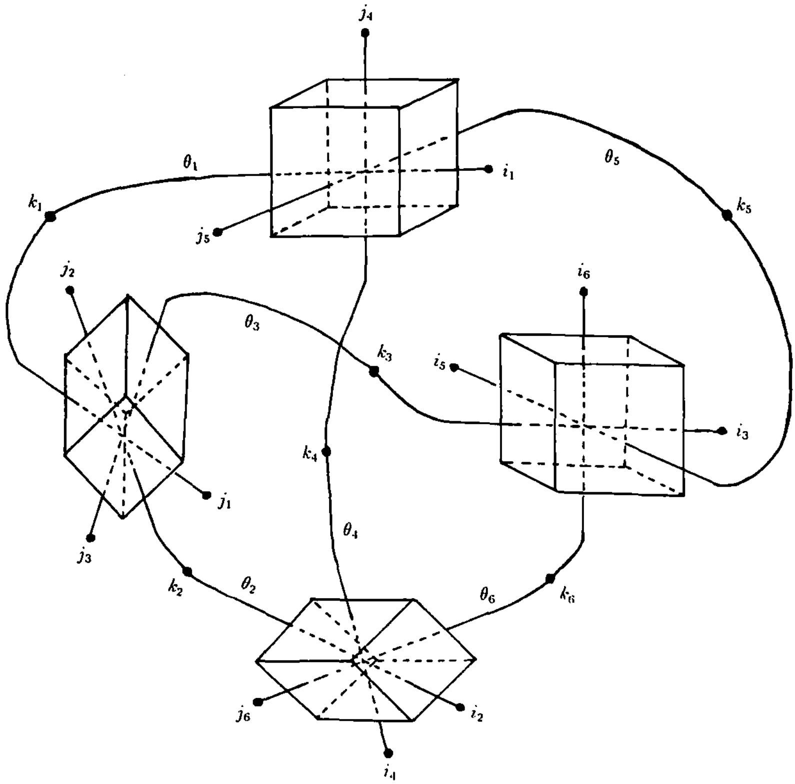

Fig. 4. The Right-hand sides of the tetrahedron equations.

where

$$
\begin{aligned}
&u_1=\frac{x_1}{\omega x_2}=\frac{x_1'}{\omega x_2'}, && u_2=\frac{x_{13}x_{24}}{\omega x_{14}x_{23}}=\frac{x_1''}{\omega x_2''}, && u_3=\frac{x_4}{x_3}=\frac{x_1'''}{\omega x_2'''} \\
&u_4=\frac{x_{13}'x_{24}'}{\omega x_{14}'x_{23}'}=\frac{x_{13}''x_{24}''}{\omega x_{14}''x_{23}''}, && u_5=\frac{x_4'}{x_3'}=\frac{x_{13}'''x_{24}'''}{\omega x_{14}'''x_{23}'''}, && u_6=\frac{x_4''}{x_3''}\times\frac{x_4'''}{x_3'''}
\end{aligned}
\tag{31}
$$

The other constraints on the spectra will be discussed in Section 5. When $u=v$, $z_1\to\infty$, $z_2\to 0$, which corresponds to the case of $\theta_1=\theta_3$, $\theta_2=0$ (see the following section), we have that[9]

$$R_{i_1 i_2 i_3}^{j_1 j_2 j_3}=(-)^{j_2+j_3}\,\delta_{j_1,j_3}$$

from relation (29). We can think of each side of the cube-type tetrahedron equation as the partition function of the four skewed cubes joined together with a common interior spin d, which forms a rhombic dodecahedron. In this way, we can express both types of tetrahedron equations graphically as in Figs. 3 and 4. These figures give also the duality between the cube-type and vertex-type tetrahedron equations.

The weight function $R_{i_1 i_2 i_3}^{j_1 j_2 j_3}$ has the symmetry property

$$R_{i_1 \pm N i_2 \pm N i_3 \pm N}^{j_1 \pm N j_2 \pm N j_3 \pm N} = R_{i_1 i_2 i_3}^{j_1 j_2 j_3} \tag{32}$$

When $N = 2$, it relates to the sublattice symmetry properties

$$\begin{aligned} W(-a \mid e, f, g \mid -b, -c, -d \mid h) &= W(a \mid -e, -f, -g \mid b, c, d \mid -h) \\ &= W(a \mid efg \mid bcd \mid h) \end{aligned} \tag{33}$$

proposed by Baxter[3] for the Zamolodchikov model. Notice that the spins $a, b, \ldots, h$ take the values ± 1 in the above relation. Indeed, Sergeev *et al.*[29] noticed that the weight function of the Baxter–Bazhanov model is almost the vertex weight function when they proved the tetrahedron equation of this model. It is interesting that the $\omega^{1/2}$-factors in the weight function of the Baxter–Bazhanov model can be written into the form $(\omega^{1/2})^{j_1 j_2 + j_2 j_3 + j_1 j_3}$. Then the three-dimensional vertex model is constructed in correspondence to the Baxter–Bazhanov model.

4. SPECTRAL PARAMETRIZATION BY USING SPHERICAL TRIGONOMETRY

In this section we parametrize the spectra of the Boltzmann weights as the dihedral angles between the "rapidity planes" passing the cubes, similar to the Zamolodchikov model.[1] Following the methods in refs. 10 and 9, we introduce a large sphere (its radius is much larger than the size of the tetrahedra) with a point near the vertices as the center. Consider four great circles on the sphere corresponding to the four "would planes."[1] A fragment of the steriographic projection of this sphere is shown in Fig. 5. Note that our angles differ from Zamolodchikov's. Define

$$\begin{aligned} &l_1 = l_{23}/N, \qquad l_2 = l_{13}/N, \qquad l_3 = l_{12}/N \\ &l'_1 = l_{45}/N, \qquad l'_2 = l_{15}/N, \qquad l'_3 = l_{14}/N \\ &l''_1 = l_{46}/N, \qquad l''_2 = l_{26}/N, \qquad l''_3 = l_{24}/N \\ &l'''_1 = l_{56}/N, \qquad l'''_2 = l_{36}/N, \qquad l'''_3 = l_{35}/N \end{aligned} \tag{34}$$

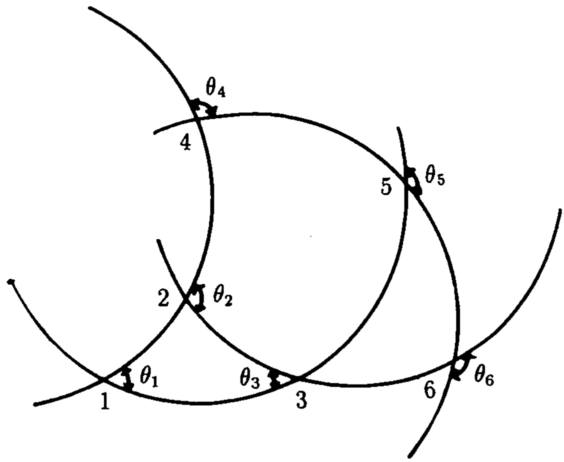

Fig. 5. A fragment of the stereographic projection of the sphere with four great circles.

where l_{ij} $(i, j = 1,..., 6,\ i < j)$ denotes the length of the segment between i and j along the circle. Then we can write

$$\begin{aligned} &x_1 = c_1/s_1, && x_2 = \omega^{-1/2} s_1/c_1 \\ &x_3 = \exp(-il_2) s_3/c_3, && x_4 = \omega^{-1/2} \exp(-il_2) c_3/s_3 \\ &x_{12} = 1/(c_1 s_1), && x_{13} = \exp[i(l-l_2)] c_2/(c_3 s_1) \\ &x_{14} = \exp[i(l_3 - l)] s_2/(s_1 s_3), && x_{23} = \omega^{-1/2} \exp[i(l_1 - l)] s_2/(c_1 c_3) \\ &x_{24} = \exp(-il) c_2/(s_3 c_1), && x_{34} = \exp(-il_2)/(c_3 s_3) \end{aligned} \tag{35}$$

The primes added to the x's are correspond to those of the c_i, s_i, l_i, l, with

$$\begin{aligned} &l = (l_{12} + l_{13} + l_{23})/(2N), && l' = (l_{14} + l_{15} + l_{45})/(2N) \\ &l'' = (l_{24} + l_{26} + l_{46})/(2N), && l''' = (l_{35} + l_{36} + l_{56})/(2N) \end{aligned} \tag{36}$$

$$\begin{aligned} &c'_1 = c_1 = [\cos(\theta_1/2)]^{1/N}, && s'_1 = s_1 = [\sin(\theta_1/2)]^{1/N} \\ &c''_1 = c_2 = [\cos(\theta_2/2)]^{1/N}, && s''_1 = s_2 = [\sin(\theta_2/2)]^{1/N} \\ &c'''_1 = c_3 = [\cos(\theta_3/2)]^{1/N}, && s'''_1 = s_3 = [\sin(\theta_3/2)]^{1/N} \\ &c''_2 = c'_2 = [\cos(\theta_4/2)]^{1/N}, && s''_2 = s'_2 = [\sin(\theta_4/2)]^{1/N} \\ &c'''_2 = c'_3 = [\cos(\theta_5/2)]^{1/N}, && s'''_2 = s'_3 = [\sin(\theta_5/2)]^{1/N} \\ &c'''_3 = c''_3 = [\cos(\theta_6/2)]^{1/N}, && s'''_3 = s''_3 = [\sin(\theta_6/2)]^{1/N} \end{aligned} \tag{37}$$

Here the angle variable $\theta_1, \theta_2, ..., \theta_6$ are chosen so that they correspond to the six spaces of the vertex-type tetrahedron equation, respectively, and differ from those used by Kashaev *et al.*[10] The parameters v_1, v_2, v_3, v_4 in Eq. (2) can be denoted by

$$v_1 = \omega^{-1} e^{il_2} s_1 s_3/(c_1 c_3), \qquad v_2 = \omega^{-1/2} e^{il_2} s_1 c_3/(c_1 s_3)$$

$$v_3 = e^{-il_2} s_1 s_3/(c_1 c_3), \qquad v_4 = \omega^{-1/2} e^{-il_2} s_1 c_3/(c_1 s_3)$$

which is just relation (2.13) provided by Bazhanov and Baxter[9] when we make the substitutions $\theta_1 \to \theta_2$, $\theta_2 \to \pi - \theta_3$, $\theta_3 \to \theta_1$.

In this way, we have

$$u_i = \omega^{-1/2} [\operatorname{ctg}(\theta_i/2)]^{2/N}, \qquad i = 1, 2, ..., 6 \tag{38}$$

The vertex-type tetrahedron equation has the form (see Figs. 3 and 4)

$$\begin{aligned} &\sum_{\substack{\{k_i\},\\ i=1,...,6}} R(\theta_1, \theta_2, \theta_3)^{k_1, k_2, k_3}_{i_1, i_2, i_3} R(\theta_1, \theta_4, \theta_5)^{j_1 k_4 k_5}_{k_1 i_4 i_5} \\ &\qquad \times R(\theta_2, \theta_4, \theta_6)^{j_2 j_4 k_6}_{k_2 k_4 i_6} R(\theta_3, \theta_5, \theta_6)^{j_3 j_5 j_6}_{k_3 k_5 k_6} \\ &\quad = \sum_{\substack{\{k_i\},\\ i=1,...,6}} R(\theta_3, \theta_5, \theta_6)^{k_3, k_5, k_6}_{i_3, i_5, i_6} R(\theta_2, \theta_4, \theta_6)^{k_2 k_4 j_6}_{i_2 i_4 k_6} \\ &\qquad R(\theta_1, \theta_4, \theta_5)^{k_1 j_4 j_5}_{i_1 k_4 k_5} R(\theta_1, \theta_2, \theta_3)^{j_1 j_2 j_3}_{k_1 k_2 k_3} \end{aligned} \tag{39}$$

with the angles satisfying the condition

$$\begin{aligned} &\left[\sin\frac{\theta_1 + \theta_2 + \theta_3}{2} \sin\frac{-\theta_1 + \theta_2 + \theta_3}{2} \sin\frac{-\theta_3 + \theta_5 + \theta_6}{2} \sin\frac{\theta_3 + \theta_5 - \theta_6}{2} \right]^{1/2} \\ &\quad - \left[\sin\frac{\theta_1 - \theta_2 + \theta_3}{2} \sin\frac{\theta_1 + \theta_2 - \theta_3}{2} \sin\frac{\theta_3 - \theta_5 + \theta_6}{2} \sin\frac{\theta_3 + \theta_5 + \theta_6}{2} \right]^{1/2} \\ &\quad = \sin\theta_3 \left[\sin\frac{\theta_2 + \theta_4 - \theta_6}{2} \sin\frac{-\theta_2 + \theta_4 + \theta_6}{2} \right]^{1/2} \end{aligned} \tag{40}$$

(see Fig. 5). This relation can be obtained from Eq. (3.2) of ref. 1 by a proper choice of the angles: $\theta_2 \to \theta_3$, $\theta_3 \to \pi - \theta_2$, $\theta_5 \to \theta_6$, $\theta_6 \to \pi - \theta_5$. There are six summation indices in the vertex-type tetrahedron equation, and only one in the cube-type tetrahedron equation. They are consistent due to the constraints $i_1 - j_1 = j_2 - i_2 = i_3 - j_3$ in the vertex-type weight function $R^{j_1 j_2 j_3}_{i_1 i_2 i_3}$. It is easy to interpret the duality between the N^{14}

cube-type tetrahedron equation (8) and the N^{12} vertex-type tetrahedron equation (39) by taking account of the symmetry properties (15) and (16). The vertex-type weight function also has the property

$$R(\theta_1, \theta_2, \theta_3)^{j_1 j_2 j_3}_{i_1 i_2 i_3} = R(\theta_1, \theta_2, \theta_3)^{i_1 i_2 i_3}_{j_1 j_2 j_3} \tag{41}$$

It corresponds to the "diagonal reversal" property

$$W(a \mid efg \mid bcd \mid h) = W(h \mid bcd \mid efg \mid a) \tag{42}$$

given by Baxter[3] for the Zamolodchikov model when $N=2$. The other properties of the weight functions will be given in the following section. The angle variable $\theta_1, \theta_2, \ldots, \theta_6$ relate the six spaces in which the vertex-type tetrahedron equation is defined.

5. SYMMETRY PROPERTIES OF THE VERTEX-TYPE WEIGHT FUNCTION

In this section first we consider the additional constraints imposed on the tetrahedron equations given by Kashaev *et al.*, from the point of view of the above angle variables. Then we find that the Boltzmann weights are symmetrical under the transformations of the group G consisting of various rotations, reflections, and their combinations of the cube with respect to the vertex-type weight functions. It can be checked easily that the angle parametrization satisfies the condition which ensures that all the similarity transformation factors[10] cancel each other. In terms of the "coordinated" parameters the four additional constraints have the form

$$\begin{aligned} &\omega \frac{x_{23}}{x_3} \frac{x'_4}{x'_{24}} \frac{x''_{24}}{x''_2} \frac{x'''_2}{x'''_{24}} = 1, \qquad && \frac{x_{13}}{x_1} \frac{x'_1}{x'_{14}} \frac{x''_{14}}{x''_1} \frac{x'''_1}{x'''_{14}} = 1 \\ &\frac{x_{14}}{x_4} \frac{x'_4}{x'_{14}} \frac{x''_{14}}{x''_4} \frac{x'''_4}{x'''_{24}} = 1, \qquad && \frac{x_{13}}{x_3} \frac{x'_3}{x'_{13}} \frac{x''_{13}}{x''_1} \frac{x'''_2}{x'''_{23}} = 1 \end{aligned} \tag{43}$$

By taking account of the expressions (28)–(30), we can change the above constraints into

$$\begin{aligned} &l_{12} + l_{24} = l_{14}, \qquad && l_{13} + l_{35} = l_{15} \\ &l_{23} + l_{36} = l_{26}, \qquad && l_{45} + l_{56} = l_{46} \end{aligned} \tag{44}$$

As shown in Fig. 5, they hold naturally. We guess that a similar geometric interpretation exists for the IRC model with the modified tetrahedron

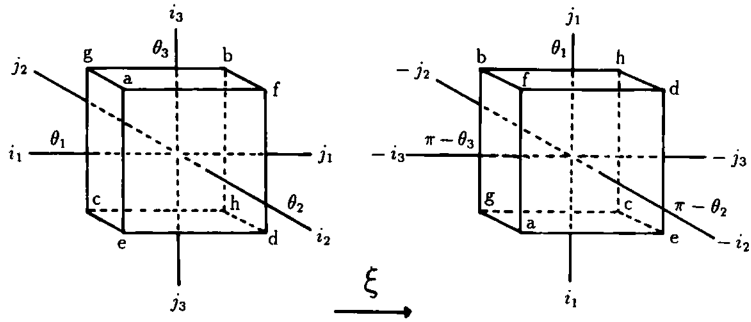

Fig. 6. The transformation ξ corresponding to the three-dimensional star–star relation.

equation.[16, 18] From refs. 9 and 11 we know that the three-dimensional star–star relation means the transformation ξ:

$$W(a \mid efg \mid bcd \mid h) \xrightarrow{\xi} W(f \mid adb \mid hge \mid c) \tag{45}$$

with

$$\frac{x_3}{x_1} \xrightarrow{\xi} \frac{x_2}{x_3}, \quad \frac{x_4}{x_2} \xrightarrow{\xi} \frac{x_1}{\omega x_4}, \quad \frac{x_4}{x_1} \xrightarrow{\xi} \frac{x_1}{\omega x_3}, \quad \frac{x_3}{\omega x_2} \xrightarrow{\xi} \frac{x_2}{\omega x_4} \tag{46}$$

In terms of the vertex form the star–star relation can be expressed as

$$R(\theta_1, \theta_2, \theta_3)^{j_1 j_2 j_3}_{i_1 i_2 i_3} = R(\pi-\theta_3, \pi-\theta_2, \theta_1)^{-j_3 - j_2 i_1}_{-i_3 - i_2 j_1} \tag{47}$$

as in Fig. 6. Under the transformations τ and ρ of the generating elements of the group G the weight functions change as[16]

$$W(a \mid efg \mid bcd \mid h) \xrightarrow{\tau} W(a \mid feg \mid cbd \mid h) \tag{48}$$

with

$$\frac{x_2}{x_1} \xleftrightarrow{\tau} \frac{x_3}{\omega x_4}, \qquad \frac{x_4}{x_1} \xleftrightarrow{\tau} \frac{x_3}{\omega x_2} \tag{49}$$

and

$$W(a \mid efg \mid bcd \mid h) \xrightarrow{\rho} W(g \mid cab \mid fhe \mid d) \tag{50}$$

with

$$\frac{x_3}{x_1} \xrightarrow{\rho} \frac{x_{13}x_4}{x_3x_{14}}, \quad \frac{x_3}{x_2} \xrightarrow{\rho} \frac{\omega x_{23}x_4}{x_3x_{24}}, \quad \frac{x_4}{x_2} \xrightarrow{\rho} \frac{x_{23}}{x_{24}}, \quad \frac{x_{14}x_{23}}{x_{13}x_{24}} \xrightarrow{\rho} \frac{x_2}{x_1} \tag{51}$$

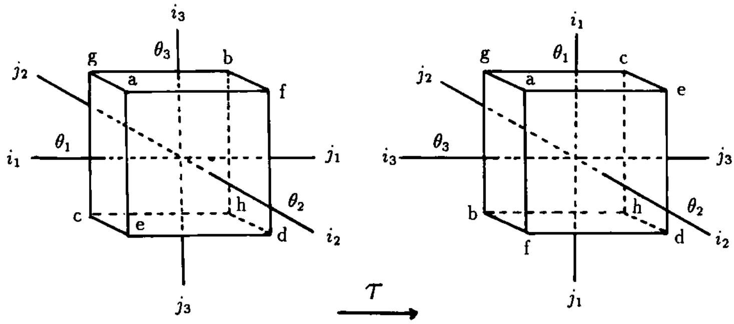

Fig. 7. The transformation τ of the weight function.

So their vertex forms are

$$R(\theta_1, \theta_2, \theta_3)^{j_1 j_2 j_3}_{i_1 i_2 i_3} = R(\theta_3, \theta_2, \theta_1)^{j_3 j_2 j_1}_{i_3 i_2 i_1} \tag{52}$$

for the transformation τ, as in Fig. 7, and

$$R(\theta_1, \theta_2, \theta_3)^{j_1 j_2 j_3}_{i_1 i_2 i_3} = R(\pi - \theta_2, \theta_1, \pi - \theta_3)^{-i_2 j_1 -j_3}_{-j_2 i_1 -i_3} \tag{53}$$

for transformation ρ, as in Fig. 8. The angles $\theta_1, \theta_2, \ldots, \theta_6$ can be interpreted as the dihedral angles between the rapidity planes connected with the cubes. With respect of the vertex model, these angles can be regarded as the parameters related to the spaces on which the vertex type tetrahedron equation is defined. So these parameters and spin variables should be transformd "regularly" under the symmetry group G. This is entirely consistent with the above equations. The geometric considerations are shown in Figs. 6–8. The above two relations are the "elementary" relations. The

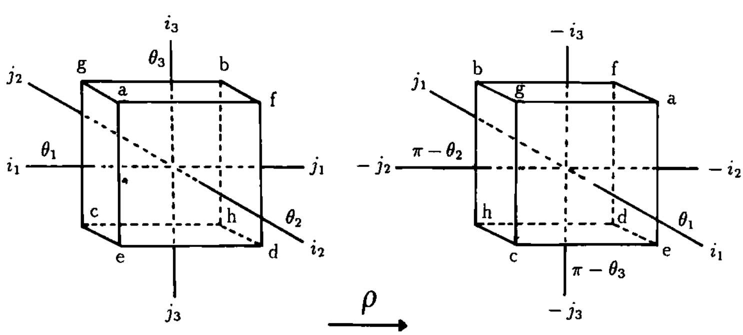

Fig. 8. The transformation ρ of the weight function.

other relations of the transformations of G can be obtained from them. It can be chcked easily that the star–star relation (47) can be obtained from relations (52) and (53).

6. SUMMARY

We obtained the duality between the cube-type weight functions and the vertex-type weight functions explicitly for the three-dimensional Baxter–Bazhanov model and found that the Boltzmann weight of the model depends on the four spin variables which are the linear combinations of the spins located on the corner sites of the cube. We interpreted the vertex-type weight function $R(\theta_1, \theta_2, \theta_3)_{i_1 i_2 i_3}^{j_1 j_2 j_3}$ as the Boltzmann weight of a three-dimensional vertex model, and the spectra $\theta_1, \theta_2, \theta_3$ were connected to the "lines" 1, 2, and 3 (see Fig. 2). In this way, we were able to write the symmetrical relations of the vertex-type Boltzmann weights in terms of the angles. We gave symmetry properties for the weight functions of the Baxter–Bazhanov model which are important for constructing the vertex model and for obtaining the duality between the two kinds of tetrahedron equations. One of the symmetry properties of the three-dimensional vertex model is related to the sublattice symmetry properties proposed by Baxter[3] when $N=2$. The angles $\theta_1, \theta_2, \theta_3$ are the dihedral angles between the rapidity planes connected with the cubes. Then the weight functions should be transformed "regularly" under the actions of the symmetry group G which consists of various rotations, reflections, and their combinations of the cube. The relations (47), (52), and (3) are entirely consistent with this (see Figs. 6–8). The angles θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 with relation (40) can be interpreted as the spectra related to the six spaces in which the vertex tetrahadron equation (39) is defined (see Figs. 3 and 4). When we set $i_3 \equiv j_3 \equiv 0$ and take appropriate spectral parameters, the Boltzmann weight of the vertex model proposed in ref. 22 can be obtained from the weight function (24) with the spin assignments.[23] Here we have 16 nonzero weights $R(\theta_1, \theta_2, \theta_3)_{i_1 i_2 i_3}^{j_1 j_2 j_3}$ for $N=2$. So it is an interesting question to find the connection between it and the solutions in ref. 18. Since the IRC model proposed in refs. 16 and 28 has similar spin assignments in the weight function to those for the Baxter–Bazhanov model, a new three-dimensional vertex model also can be constructed[29] as a duality of the new series of the three-dimensional integrable lattice models with N colors, where the weight functions satisfy the modified vertex-type tetrahedron equation. We guess that a geometric interpretation exists also for the constraint conditions of the modified tetrahedron equation as a kind of deformation of those in the Baxter–Nazhanov model.

The three-dimensional Baxter–Bazhanov model was built through the study of the chiral Potts model, which is a generalization of the free-fermion model owing to the work of Baxter *et al.*[30] In this formulation, their work earlier provided the two-layer Baxter–Bazhanov model. The chiral Potts model can be regarded as a descendant of the six-vertex model in two dimensions.[31–33] This relation appears first, implicitly, in the Bethe-ansatz solution of Albertini *et al.*[34] and the conjectured equation was derived by Bazhanov and Stroganov.[33] In ref. 35, the chiral Potts model is discussed in detail and the funtional relations of the transfer matrices are established, which is the starting point of the most recent calculations in the chiral Potts model. Recently, the hidden symmetries in the six-vertex model and the correlation function[36,37] of the XXZ chain[38] have been discussed.[39,40] Then there are the interesting problems of the correlation function and calculating the free energy for this 3D vertex model by using the duality discussed above, as by Baxter and Bazhanov[2,9] for the Zamolodchikov model and the IRC model. The phase transitions of lattice models of interacting systems are also interesting subjects in statistical mechanics. The Properties of the phase transitions of the XXZ chain and the XYZ chain have been studied by various methods[38,41,42]: the exact solutions, the approximate approaches by the use of the Jordan–Wigner transformation,[43] and the free Fermi system.[44–47] We hope that the discussions in this paper will be useful in studying the critical behaviors of lattice models of interacting systems in three dimensions.[49] But it must be noted that the weights of the Baxter–Bazhanov model are not real and positive, apart from the original chiral Potts model (two-layer case), nor is there a Hermitian associated quantum model beyond the chain case, as in the very first work by Au-Yang *et al.*,[49] which is a major shortcoming of the Zamolodchikov–Baxter–Bazhanov model.

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REFERENCES

1. A. B. Zamolodchikov, *Commun. Math. Phys.* **79**:489 (1981).
2. R. J. Baxter, *Physica D* **18**:321 (1986).
3. R. J. Baxter, *Commun. Math. Phys.* **88**:185 (1983).
4. F. W. Wu, *Phys. Rev. B* **4**:2312 (1971).
5. L. P. Kadanoff and F. J. Wegner, *Phys. Rev. B* **4**:3989 (1971).
6. V. V. Bazhanov and R. J. Baxter, *J. Stat. Phys.* **69**:453 (1992).
7. R. M. Kashaev, V. V. Mangazeev, and Yu. G. Stroganov, *Int. J. Mod. Phys. A* **8**:1399 (1993).
8. Z. N. Hu, *Mod. Phys. Lett. B* **8**:779 (1994).
9. V. V. Bazhanov and R. J. Baxter, *J. Stat. Phys.* **71**:839 (1993).
10. R. M. Kashaev, V. V. Mangazeev, and Yu. G. Stroganov, *Int. J. Mod. Phys. A* **8**:587 (1993).
11. Z. N. Hu, *Int. J. Mod. Phys. A* **9**:5201 (1994).
12. Z. N. Hu and B. Y. Hou, *J. Stat. Phys.* **79**:759 (1995).
13. L. D. Faddeev and R. M. Kashaev, Quantum dilogarithm, Preprint (1993).
14. Z. N. Hu, *Phys. Lett. A* **197**:387 (1995).
15. Z. N. Hu and B. Y. Hou, *Phys. Lett. A* **201**:151 (1995).
16. V. V. Mangazeev, S. M. Sergeev, and Yu. G. Stroganov, *Int. J. Mod. Phys. A* **9**:5517 (1994).
17. V. V. Mangazeev and Yu. G. Stroganov, *Mod. Phys. Lett. A* **8**:3475 (1993).
18. I. G. Korepanov, *Commun. Math. Phys.* **154**:85 (1993).
19. I. Frenkel and G. Moore, *Commun. Math. Phys.* **138**:259 (1991).
20. Y. Q. Li and Z. N. Hu, *J. Phys. A* **28**:L219 (1995).
21. J. Hietarinta, *J. Phys. A* **27**:5727 (1994).
22. V. V. Mangazeev, S. M. Sergeev, and Yu. G. Stroganov, New solution of vertex type tetrahedron equations, Preprint [hep-th/9410049] (1994).
23. Z. N. Hu, On the three-dimensional lattice model, preprint (1995).
24. J. H. H. Perk and F. Y. Wu, *J. Stat. Phys.* **42**:727 (1986).
25. H. Au-Yang and J. H. H. Perk, *Adv. Stud. Pure Math.* **19**:57 (1989).
26. S. M. Sergeev, V. V. Mangazeev, and Yu. G. Stroganov, The vertex formulation of the Bazhanov–Baxter model, Preprint [hep-th/9504035] (1995).
27. S. Sergeev, Yu. Stroganov, V. Mangazeev, and G. Boos, Private communication (1995).
28. H. E. Boos, V. V. Mangazeev, and S. M. Sergeev, Modified tetrahedron equation and related 3D integrable models, Preprint [hep-th/9407146].
29. Z. N. Hu, Three-dimensional vertex model related BCC model in statistical mechanics, Preprint (1995).
30. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, *Phys. Lett. A* **128**:138 (1988).
31. E. H. Lieb, *Phys. Rev.* **162**:162 (1967); *Phys. Rev. Lett.* **18**:692, 1046 (1967); *Phys. Rev. Lett.* **19**:108 (1967).
32. M. T. Batchelor, R. J. Baxter, M. J. O'Rourke, and C. M. Yung, Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions, Preprint, Australian National University MRR 012-95 (1995).
33. V. V. Bazhanov and Yu. G. Stroganov, *J. Stat. Phys.* **59**:799 (1990).
34. G. Albertini, B. M. McCoy, and J. H. H. Perk, *Phys. Lett. A* **135**:159 (1989); *Adv. Stud. Pure Math.* **19**:1 (1989).
35. R. J. Baxter, V. V. Bazhanov, and J. H. H. Perk, *Int. J. Mod. Phys. B* **4**:803 (1990).
36. M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki, *Phys. Lett. A* **168**:256 (1992).
37. N. S. Han and A. Honecker, *J. Phys. A* **27**:9 (1994).

38. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
39. I. G. Korepanov, Hidden symmetries in the 6-vertex model of statistical physics, Preprint (1994).
40. F. H. L. Essler, H. Frahm, A. R. Its, and V. E. Korepin, Integro-difference equation for a correlation function of the spin-$\frac{1}{2}$ Heisenberg *XXZ* chain, Preprint, BONN-H-95-06 (1995).
41. J. D. Johnson, S. Krinsky, and B. M. McCoy, *Phys. Rev. A* **8**:2526 (1973).
42. Z. J. Chen, Y. M. Zhang, and B. W. Xu, *Z. Phys. B*, to appear.
43. D. Jordan and R. Wigner, *Z. Phys.* **47**:631 (1928).
44. V. J. Emery, In *Highly Conducting One-Dimensional Solids*, J. T. Devreese, R. P. Evrard, and V. E. van Doren, eds. (Plenum Press, New York, 1979), p. 247.
45. K. M. Leung, *Phys. Rev. B* **27**:2877 (1983).
46. E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, Massachusetts, 1991).
47. V. V. Bazhanov and Yu. G. Stroganov, *Teor. Mat. Fiz.* **62**:377 (1985).
48. R. J. Baxter and P. J. Forrester, *J. Phys. A* **18**:1483 (1985).
49. H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. L. Yan, *Phys. Lett. A* **123**:219 (1987).
50. B. L. Cerchiai, M. Martellini, and F. Valz-Gris, *J. Stat. Phys.* **78**:1083 (1995).
51. S. Boukraa, J.-M. Maillard, and G. Rollet, *J. Stat. Phys.* **78**:1195 (1995).